Change of numéraire (Cont)

Math 622

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1 The exchange rate

Recall the exchange rate model. There is asset price $S(t)$, foreign exchange rate $Q(t)$, domestic money market rate $R(t)$ and foreign money market rate $R^f(t)$. Recall

$$N^f(t) = \exp\{\int_0^t R^f(u)\,du\}Q(t) \tag{1}$$

is the dollar value of one unit of the foreign money market account. The risk-neutral model when prices are in dollars is

$$\begin{aligned} dS(t) &= R(t)S(t)\,dt + \sigma_1(t)S(t)\,d\widetilde{W}_1(t) \\ dN^f(t) &= R(t)N^f(t)\,dt + N^f(t)\sigma_2(t)\left[\rho(t)\,d\widetilde{W}_1(t) + \sqrt{1-\rho^2(t)}\,d\widetilde{W}_2(t)\right] \end{aligned} \tag{2}$$

The risk-neutral measure $\widetilde{\mathbf{P}}^{(N^f)}$ when prices are denominated using N^f as numéraire is given in Shreve, page 386, equation (9.3.17) and in the previous set of lecture notes. Here, we make some remarks concerning the exchange rate process $Q(t)$.

1.1 The exchange rate under the domestic risk-neutral measure

It follows from equations (1), (2) that

$$dQ(t) = [R(t) - R_f(t)]\,Q(t)\,dt + Q(t)\sigma_2(t)\left[\rho(t)\,d\widetilde{W}_1(t) + \sqrt{1-\rho^2(t)}\,d\widetilde{W}_2(t)\right] \tag{3}$$

When dealing with Q alone it is convenient to write this in a simpler form. Define

$$\widetilde{W}_3(t) = \int_0^t \left[\rho(t)\,d\widetilde{W}_1(t) + \sqrt{1-\rho^2(t)}\,d\widetilde{W}_2(t)\right].$$

Observe that

$$\begin{aligned} [d\widetilde{W}_3(t)]^2 &= \left[\rho(t)\, d\widetilde{W}_1(t) + \sqrt{1-\rho^2(t)}\, d\widetilde{W}_2(t)\right]^2 \\ &= \rho^2(t)\, dt + (1-\rho^2(t))\, dt = dt. \end{aligned}$$

Then by Itô's rule,

$$\widetilde{W}_3^2(t) = \int_0^t \widetilde{W}_3(u) \left[\rho(t)\, d\widetilde{W}_1(t) + \sqrt{1-\rho^2(t)}\, d\widetilde{W}_2(t)\right] + t.$$

Hence $\widetilde{W}_3^2(t) - t$ is a martingale. $\widetilde{W}_3(t)$ is also a continuous martingale starting at 0, and so Lévy's theorem implies that $\widetilde{W}_3(t)$ is itself a Brownian motion. Using $\widetilde{W}_3$,

$$dQ(t) = [R(t) - R_f(t)]\, Q(t)\, dt + \sigma_2(t) Q(t)\, d\widetilde{W}_3(t). \tag{4}$$

Remark: The foreign exchange rate behaves exactly like a risky asset that pays dividents at rate $R^f(t)$. Equation (4) is the same as equation (5.5.6) in Shreve for a dividend-paying asset if $A(t)$ in that equation is replaced by $R^f(t)$.

1.2 Black-Scholes formula for a Call option on the exchange rate

Let $\sigma_2(t) = \sigma_2$ be constant, and also let $R(t) = r$ and $R^f(t) = r^f$ be constant. Then equation (4) becomes

$$dQ(t) = \left[r - r^f\right] Q(t)\, dt + \sigma_2 Q(t)\, d\widetilde{W}_3(t). \tag{5}$$

The solution to this equation is

$$Q(t) = Q(0) \exp\{\sigma_2 \widetilde{W}_3(t) + (r - r^f - \frac{1}{2}\sigma^2)t\}. \tag{6}$$

We can look at $Q(t)$ (*from a computational point of view*) as the Black-Scholes price of an asset following the geometric Brownian motion model, when the volatility is σ_2 and *the risk free rate is* $r^f - r$.

The fact that $Q(t)$ is a classical Black-Scholes price gives immediate formulas for options on the exchange rate in the constant coefficient case, which we will develop below.

Suppose that the risk free rate is r and under $\tilde{P}$, a stock S_t has dynamics:

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t.$$

Let $C(T-t,x,K,r,\sigma)$ the price of at time t of a European call on S with strike K, conditioned on $S_t = x$. That is

$$C(T-t,x,K,r,\sigma) = \tilde{E}\Big(e^{-r(T-t)}(S_T-K)^+\Big|S_t = x\Big).$$

Then the Black-Scholes formula for $C(T-t,x,K,r,\sigma)$ is

$$\begin{aligned}
C(T-t,x,K,r,\sigma) &= e^{-r(T-t)}\tilde{E}\left[\left(xe^{\sigma\widetilde{W}(T-t))+(r-\sigma^2/2)(T-t)} - K\right)^+\right] \\
&= xN\left(\frac{\ln(x/K)+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad - Ke^{-r(T-t)}N\left(\frac{\ln(x/K)+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}$$

Consider now a European call option on $Q(T)$ at strike K, for the model of (6). This can also be looked at as a call option with strike K on *a unit of foreign currency, quoted in domestic currency.*

According to risk-neutral pricing, the value of this option at time t is

$$\begin{aligned}
V(t) &= e^{-r(T-t)}\tilde{E}\left[(Q(T)-K)^+ \mid \mathcal{F}(t)\right] \\
&= e^{-r^f t}e^{-(r-r^f)(T-t)}\tilde{E}\left[(Q(T)-K)^+ \mid Q(t)\right].
\end{aligned}$$

But evaluating $e^{-(r-r^f)(T-t)}\tilde{E}\left[(Q(T)-K)^+ \mid Q(t)\right]$ is exactly the same as evaluating the price of a European call when the risk free rate is $r-r^f$ and the volatility is σ_2. Therefore,

$$V(t) = e^{-r^f t}C(T-t,Q(t),K,r-r^f,\sigma_2).$$

This is called the Garman-Kohlhagen formula. You can also recover this formula from the formula (5.5.12) in Shreve for the price of a call on dividend-paying asset. Just replace a in this formula by r^f.

1.3 The exchange rate from the foreign currency viewpoint

Starting with the model (1)-(2), suppose we use the foreign currency money market $N^f(t)$ as the numéraire. In the previous lecture we found that

$$\widetilde{W}^{(N^f)}(t) = (\widetilde{W}_1(t) - \int_0^t \sigma_2(u)\rho(u)\,du, \widetilde{W}_2(t) - \int_0^t \sigma_2(u)\sqrt{1-\rho^2(u)}\,du)$$

is a Brownian motion under $\widetilde{\mathbf{P}}^{(N^f)}$ and we showed

$$dS^{(N^f)}(t) = S^{(N^f)}(t)\left[(\sigma_1(t) - \sigma_2(u)\rho(u))\, d\widetilde{W}_1^{(N^f)}(t) - \sigma_2(t)\sqrt{1-\rho^2(t)}\, d\widetilde{W}_2^{(N^f)}(t)\right]$$

To completely describe the model from the viewpoint of the foreign currency we should also look at the dollar to foreign currency exchange rate $1/Q(t)$, which is the value of one dollar in units of the foreign currency. The equation for this should have a form symmetrical to the equation (5) for $Q(t)$ when units are in dollars. Indeed,

$$d\left[\frac{1}{Q(t)}\right] = [R^f(t) - R(t)]\frac{1}{Q(t)}\, dt - \sigma_2(t)\frac{1}{Q(t)}\left[\rho(u)\, d\widetilde{W}_1^{(N^f)}(t) + \sqrt{1-\rho^2(t)}\, d\widetilde{W}_2^{(N^f)}(t)\right]. \tag{7}$$

This may be verified from Itô's rule, but one can see why it must be correct by the following reasoning. From the perspective of numéraire N^f, $R^f(t)$ is the domestic risk free rate and $R(t)$ is the domestic rate, so, where $R(t) - R^f(t)$ appears in (4), $R^f(t) - R(t)$ appears in (7). The volatility terms are essentially the same because the same stochastic fluctuation is obviously driving both $Q(t)$ and $1/Q(t)$. To explain why σ_2 appears in (4) but $-\sigma_2$ appears in (7) just note that $1/Q(t)$ goes down when Q goes up and vice-versa.

Concerning this topic, the student should read section 9.3.4 on Siegel's paradox (which is not really a paradox, but arises from a misunderstanding of the correct numéraire to use in interpreting a model.)

2 Zero coupon bonds as numéraire

In this section we assume given a risk-neutral model with a stochastic interest rate process $R(t)$, $t \geq 0$.

2.1 Zero-coupon bonds

Bonds are financial instruments that promise fixed payoffs. Most bonds provide periodic payments called coupons and then a final payment consisting of a coupon and a lump sum called the *principal* or *face value*. A zero-coupon bond pays out only at the terminal time. We let $B(t,T)$ denote the price at time $t \leq T$ of a zero-coupon bond that pays \$1 at time T.

Given a risk-neutral model defined by a probability measure $\widetilde{\mathbf{P}}$, the no-arbitrage principle demands that $D(t)B(t,T)$ be a martingale in t up to time T. Since $B(T,T) = 1$, it follows that

$$B(t,T) = \frac{\tilde{E}[D(T)B(T,T) \,\big|\, \mathcal{F}(t)]}{D(t)} = \frac{\tilde{E}[D(T) \,\big|\, \mathcal{F}(t)]}{D(t)} = \frac{\tilde{E}\left[e^{-\int_0^T R(u)\,du} \,\Big|\, \mathcal{F}(t)\right]}{e^{\int_0^t R(u)\,du}}. \tag{8}$$

Hence,

$$B(t,T) = \tilde{E}\left[e^{-\int_t^T R(u)\,du} \,\Big|\, \mathcal{F}(t)\right] \tag{9}$$

This is an interesting formula. If $R(\cdot)$ is a random process, and we are at time t, we do do not know what R will be exactly after time t. But we do the market tells us what all zero-coupon bond prices are. Any model we create for R must be consistent with (in quant lingo, must be calibrated to) the zero-coupon bond prices via (9).

2.2 Forward prices

Suppose at time t, where $t < T$, Alice contracts to buy a unit of an asset from Bob at price F at time T. This is called a forward contract. No money changes hands at time t. Let $S(u)$ denote the price of the asset as a function of time u. From Alice's perspective she is getting an option that pays off $S(T) - F$, because she is purchasing something worth $S(T)$ dollars for F dollars at time T. The value of this option at t is $D^{-1}(t)\tilde{E}[D(T)(S(T) - F) \,\big|\, \mathcal{F}(t)] = S(t) - FB(t,T)$; remember, $D(t)S(t)$ is a martingale with respect to $\widetilde{\mathbf{P}}$! If she is paying or receiving no money for the contract at time t this value should be zero. Hence

$$F = \frac{S(t)}{B(t,T)}$$

is the fair price for this contract. It is called the T-forward price and denoted by $\mathrm{For}_S(t,T)$. Really, it is the price of $S(t)$ obtained using $B(t,T)$ as a numéraire.

A trivial but important observation is that the forward price and the market price concur at time T:

$$\mathrm{For}_S(T,T) = \frac{S(T)}{B(T,T)} = S(T). \tag{10}$$

2.3 The risk-neutral measure associated with the zero-coupon bond

Under the domestic risk neutral measure $\tilde{P}$, $D_t B(t,T)$ is a martingale. Therefore, $B(t,T)$ can be used as a numéraire. Indeed, the risk-neutral measure corresponding to numéraire $B(t,T)$, according to Theorem 3 of Lecture 9, is

$$\widetilde{\mathbf{P}}^T(A) = \tilde{E}[\mathbf{1}_A \frac{D(T)B(T,T)}{B(0,T)}] = \frac{1}{B(0,T)}\tilde{E}[\mathbf{1}_A D(T)]$$

We will call P^T, following Shreve (Definition 9.4.1), *the T-forward measure.*

Consider the special case in which the filtration in the risk neutral market is generated by a single Brownian motion $\widetilde{W}$. Then in this case we know from Theorem 9.1 of Shreve that there is a process $\nu_T(u)$ such that

$$\frac{D(T)}{B(0,T)} = e^{\int_0^T \nu_T(u)\, d\widetilde{W}(u) - \frac{1}{2}\int_0^T \nu_T^2(u)\, du}$$

and that $\widetilde{W}^T(t) = \widetilde{W}(t) - \displaystyle\int_0^t \nu_t(u)\, du$ is a Brownian motion under $\widetilde{\mathbf{P}}^T$. (In Shreve, 9.4.2, the notation $-\sigma^*(t,T)$ stands for our $\nu_T(t)$.

2.4 Pricing under the T-forward measure

2.4.1 Pricing under the domestic risk neutral measure with random interest rate

Suppose the interest rate is R_t, an adapted process. Then the risk neutral price V_t of a Euro style financial product that pays V_T at time T is

$$V_t = \tilde{E}\Big(e^{-\int_t^T R_u du} V_T \Big| \mathcal{F}_t\Big).$$

Since all we know about R_t is that it is an adpated process, we cannot go further with this pricing formula, unless we make some assumption on R_t (which is about modeling the interest rate, the topic of next Chapter). This is certainly a complex topic. Moreover, even if we have a model for R_t, it doesn't mean the pricing formula will be simple, if $\int_t^T R_u du$ has non zero correlation with V_T, for example. However, a nice observation here is that we do not have to compute this equation under $\tilde{E}$. Indeed, recall from the section 5.2 result, we have:

$$V_t^T = \tilde{E}^T\Big(V_T^T \Big| \mathcal{F}_t\Big).$$

where $V_t^T := \frac{V_t}{B(t,T)}$ is the price of the product denoted in the unit of zero-coupon bond. Note that since $B(T,T) = 1$, we have $V_T^T = V_T$.

The nice thing about the pricing formula under $\tilde{P}^T$ is that it is only a conditional expectation of the terminal value, not involving other quantities like the interest rate (this is not a pure gain, since the interest rate was absorbed into $\tilde{E}^T$). However, this suggests a new approach to the entire problem: we may directly model the assets under $\tilde{P}^T$, rather than under domestic measure $\tilde{P}$. Note that if we model the asset under $\tilde{P}^T$, then the unit of denomination (or the numéraire) is the price of zero coupon bond $B(t,T)$. In particular, if our objective is to model the stock price S_t (under $\tilde{P}$) then under $\tilde{P}^T$, we model

$$S_t^T := \frac{S_t}{B(t,T)} = For_S(t,T).$$

Pricing a call option on $S(t)$ under the domestic risk neutral measure is equivalent to pricing a call option on the forward price $For_S(t,T)$ under the T-forward measure. The advantage here is again about modeling. If we model under $\tilde{P}$ then necessarily we need to involve the model of R_t and need to know how to handle the expectation $\tilde{E}\left(e^{-\int_t^T R_u du} V_T \middle| \mathcal{F}_t\right)$. If we model under $\tilde{P}^T$ then we only need to model the forward price of S_t (which potentially maybe easier to calibrate to market parameters than modeling R_t) and then the expectation $\tilde{E}^T\left(V_T^T \middle| \mathcal{F}_t\right)$ is straight forward. The detailed computation is done in the following section.

2.4.2 Pricing a Call option under the T-forward measure

Here one assumes that the forward price of asset S, is given by the simple formula

$$d\mathrm{For}_S(t,T) = \sigma \mathrm{For}_S(t,T)\, d\widetilde{W}^T(t), \qquad t \leq T.$$

The point is that this is the Black-Scholes price model with $r = 0$, and if one looked at $S(t)$ under the original risk-neutral measure, it would not follow a Black-Scholes model with constant volatility. However it is possible to explicitly price a call. Indeed, let $C(T-t, x, K, r, \sigma)$ denote the Black-Scholes price of a European call when the price is x, the risk-free interest rate is r and the volatility is σ. Let $V(t)$ be the dollar price of the call. Then its forward price is $V^T(t) = V(t)/B(t,T)$. But, recalling from (10) that $\mathrm{For}_S(T,T) = S(T)$, we know from risk-neutral pricing that

$$\begin{aligned} V^T(t) &= \tilde{E}^T\left[\frac{(S(T)-K)^+}{B(T,T)} \;\middle|\; \mathcal{F}(t)\right] = \tilde{E}^T\left[(\mathrm{For}_S(T,T)-K)^+ \;\middle|\; \mathcal{F}(t)\right] \\ &= \tilde{E}^T\left[(\mathrm{For}_S(T,T)-K)^+ \;\middle|\; \mathrm{For}_S(t,T)\right]. \end{aligned}$$

But since $\mathrm{For}_S(t,T)$ follows the Black-Scholes price model with $r = 0$ and volatility σ,

$$V^T(t) = C(T-t, \mathrm{For}_S(t,T), K, 0, \sigma).$$

Hence,

$$V(t) = B(t,T)C(T-t, \mathrm{For}_S(t,T), K, 0, \sigma).$$

By substitution into the explicit formula for C (given above on page 2),

$$\begin{aligned} V(t) &= B(t,T)\mathrm{For}_S(t,T)N\left(\frac{\ln(\frac{\mathrm{For}_S(t,T)}{K}) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\quad - KB(t,T)N\left(\frac{\ln(\frac{\mathrm{For}_S(t,T)}{K}) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right) \end{aligned}$$

This is essentially formula (9.4.9) in Shreve.

Clearly, this procedure could be applied to other cases where explicit pricing formulae are known for the Black-Scholes price model.